COO-3077-29
MF-70

Courant Institute of
Mathematical Sciences

Magneto-Fluid Dynamics Division

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I. A Practical Stability Concept

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AEC Research and Development Report

Plasma Physics
July 1973

New York University

UNCLASSIFIED

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A NEW APPROACH TO MAGNETOHYDRODYNAMIC STABILITY

I. A PRACTICAL STABILITY CONCEPT

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July 1973

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This work was supported by the U.S. Atomic Energy Commission under Contract No. AT(11-1)-3077. It was partly supported by Euratom, FOM, and ZWO.

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Abstract

An equilibrium is called σ-stable if growth faster than $\exp \sigma t$ does not occur. On the basis of this definition one obtains a modified energy principle by means of which the stability of plasma confinement systems can be tested for times of thermonuclear interest, instead of the infinitely long times which are pertinent to the usual stability analysis. The theory is applied to the diffuse linear pinch, a theorem for σ-stability is derived, and the connection with the normal-mode analysis is shown to be given with the Sturmian property, which holds for the unstable side of the spectrum, whereas the stable side consists of Sturmian and anti-Sturmian point spectra separated by continuous spectra. Growth rates and eigenfunctions of Suydam modes are numerically calculated and it is shown that violation of Suydam's criterion in the high and intermediate shear case leads to nonlocalized rapidly growing m=1 modes. Consequently, this criterion obtains new relevance in the σ-stability analysis for this regime.

1. Introduction

The problem of magnetohydrodynamic stability, in particular of the diffuse linear pinch, is one of the most extensively investigated subjects in plasma physics. The subject has recently found renewed interest, mainly because of its evident implications for plasma confinement experiments aiming at thermonuclear fusion. An understanding of the basic features involved in stability is clearly crucial to further progress in this field. In this and the accompanying paper we aim at a contribution towards such understanding.

Our starting point is that for thermonuclear confinement of plasma the usual stability concept is too strong. Indeed, one is not really interested in whether or not a plasma is stable, but one is interested in whether or not one can confine plasma long enough in order to obtain fusion. Whereas the usual stability concept requires stability with respect to all growing perturbations, including those which take an infinitely long time, all that is needed is only stability with respect to perturbations which grow in times shorter than some characteristic time needed for fusion. In this paper we introduce a new stability concept which systematically exploits this fact.

The concept of stability we use is that of σ-stability. In essence, we allow perturbations to grow at most like $e^{\sigma t}$, where $\sigma = 1/\tau$ if τ is some characteristic time needed for fusion. This approach to stability is not necessarily restricted to thermonuclear plasma stability problems, but can be useful for any physical problem where a dynamical system is studied for a limited period of time

only. In plasma physics, e.g., one could also choose τ to be the time during which one considers the simplified equations of ideal MHD to be a reasonable approximation of the real situation. Alternatively, for a plasma that is contained during a time which is solely determined by the decay of external circuitry, one could choose τ to be the decay time of the circuits in order to study the question of stability during that time.

A modified energy principle for σ-stability, similar to the usual energy principle which pertains to marginal stability ($\sigma = 0$), is obtained from the theory by Laval et al. [1]. It is clear that this way of treating stability problems is closely related to a normal-mode analysis, where a time-dependence $e^{i\omega t}$ is assumed and eigenvalues ω are calculated. One might even wonder whether the whole concept of σ-stability does not boil down to a normal-mode analysis. This is not the case, the important difference being that in a normal-mode analysis the eigenvalue ω has to be determined, whereas in a σ-stability analysis the value of σ is simply fixed beforehand. Hence, the problem is of the same nature as a stability analysis by means of the energy principle, although the equations are more complicated (i.e., they have more terms).

The latter disadvantage is more than offset by the fact that the σ-stability analysis avoids the difficult and, in this respect, irrelevant questions pertinent to marginal stability. The content of the difficulties associated with the marginal stability analysis have recently been expounded by Grad [2] in a spectral study of ideal MHD. For the diffuse linear pinch the spectrum in general consists of an unstable point spectrum often accumulating at $\omega^2 = 0$, a number of stable pointspectra, and a number of

stable continuous spectra frequently extending to $\omega^2 = 0$. Moreover, although there are infinitely many eigenvalues close to it, the point $\omega^2 = 0$ itself is often missing from the point spectrum. This example shows that the marginal stability analysis concentrates on that part of the spectrum which is both highly singular and highly irrelevant, so that the supposed simplicity of it often turns out to be illusory.

Equally important as the above-mentioned conceptual advantages of the σ-stability analysis are the practical implications. They will become apparent in the accompanying paper where a systematic investigation is made of the equilibrium profiles of a diffuse linear pinch that are σ-stable. The essential point is that we find many profiles which are σ-stable, but which would have been discarded under the usual definition of stability. A similar result emerges from the analysis of low-shear systems [2]. Here, the stability advantages of closed-line systems over sheared systems are transferred to the broad class of low-shear systems when one adopts a slightly different effective stability boundary associated with the onset of significant growth. This allows one to produce a rather complete classification of diffuse linear pinch equilibria, which are acceptable in fusion devices as far as ideal magnetohydrodynamics is concerned. Results for high-shear systems are given in the accompanying paper II, whereas a future third paper will be devoted to low-shear systems.

This paper is arranged as follows. In Sec. 2 we formally define σ-stability and introduce the modified energy principle giving a necessary and sufficient criterion for it. In Sec. 3

this theory is applied to the diffuse linear pinch, yielding a formulation for σ-stability analogous to Newcomb's [3] theory for marginal stability. Both formulations turn out to be based on the Sturmian property of the discrete unstable side of the spectrum, which is proved here for one-dimensional systems. This proof is general enough to permit also conclusions about the stable side of the spectrum and we show that there the spectrum consists of Sturmian and anti-Sturmian discrete spectra separated by continuous spectra. In Sec. 4 we slightly deviate from the given exposition in order to study the spectral questions connected with Suydam modes. We find very non-standard results relating to the localization of these modes. This section serves as the connection with the accompanying paper, where the problem of σ-stability of the diffuse linear pinch is studied numerically, culminating in a classification of σ-stable equilibrium profiles with the special emphasis on use for fusion and experimental accessibility.

Throughout this and the next paper we employ a rationalized Gaussian system of units with $c = 1$ or an MKS system with $\mu_o = 1$, leaving it to the preference of the reader to divide B^2 by 4π or by μ_o.

2. σ-Stability

We review the required results of magnetohydrodynamics. The equations are written as

$$\rho \frac{d\underset{\sim}{v}}{dt} = - \nabla p - \underset{\sim}{B} \times (\nabla \times \underset{\sim}{B}) \ , \tag{1}$$

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{v} \times \underset{\sim}{B}) \ , \tag{2}$$

$$\frac{\partial p}{\partial t} = - \underset{\sim}{v} \cdot \nabla p - \gamma p \nabla \cdot \underset{\sim}{v} \ , \tag{3}$$

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \underset{\sim}{v}) \ , \tag{4}$$

where the first equation can be interpreted as the equation of motion of the plasma, whereas the latter three equations are differential expressions for flux, entropy, and mass conservation [4]. We are concerned with linear stability of static equilibria described by

$$\nabla p + \underset{\sim}{B} \times (\nabla \times \underset{\sim}{B}) = 0 \ . \tag{5}$$

Perturbations of these equilibria are conveniently examined in terms of the displacement vector field $\underset{\sim}{\xi}$ of plasma elements connected with the velocity according to

$$\underset{\sim}{v} = \frac{\partial \underset{\sim}{\xi}}{\partial t} \ . \tag{6}$$

To first order in $\underset{\sim}{\xi}$ the equations (2)-(4) are then easily integrated to yield the perturbed expressions for $\underset{\sim}{B}$, p, and ρ. Substituting these into Eq. (1) the basic linearized equation of motion of ideal MHD is obtained:

$$\underset{\sim}{F}\{\underset{\sim}{\xi}\} = \rho \frac{\partial^2 \underset{\sim}{\xi}}{\partial t^2} . \tag{7}$$

Here, $\underset{\sim}{F}$ is the linear force-operator as introduced by Bernstein et al. [5]:

$$\underset{\sim}{F}\{\underset{\sim}{\xi}\} = \nabla(\gamma p \nabla \cdot \underset{\sim}{\xi} + \underset{\sim}{\xi} \cdot \nabla p) - \underset{\sim}{Q} \times (\nabla \times \underset{\sim}{B}) - \underset{\sim}{B} \times (\nabla \times \underset{\sim}{Q}) , \tag{8}$$

where

$$\underset{\sim}{Q} = \nabla \times (\underset{\sim}{\xi} \times \underset{\sim}{B}) , \tag{9}$$

and ρ, p, and $\underset{\sim}{B}$ indicate the zeroth order equilibrium quantities.

The equation of motion (7) provides a nearly complete basis for the study of linear stability, the spectrum, and the initial value problem in ideal MHD when supplemented with the proper initial and boundary values. Since the equation is of second order in time it is sufficient to specify $\underset{\sim}{\xi}(0)$ and $\dot{\underset{\sim}{\xi}}(0)$ in the domain of interest, e.g. a toroidal region completely occupied with plasma and surrounded by a perfectly conducting wall. In that case, proper boundary conditions for the sixth order spatial system are periodicity of $\underset{\sim}{\xi}$ the long and the short way around the torus, regularity at the magnetic axis, and vanishing of the normal component of $\underset{\sim}{\xi}$ at the wall. More general boundary conditions are possible, but they are usually restricted to those that leave $\underset{\sim}{F}$ self-adjoint:

$$\langle \underset{\sim}{\eta}, \underset{\sim}{F}\{\underset{\sim}{\xi}\} \rangle = \langle \underset{\sim}{\xi}, \underset{\sim}{F}\{\underset{\sim}{\eta}\} \rangle , \tag{10}$$

where the angular bracket notation for scalar products in function space is used.

Two approaches to the linear stability problem on the basis of Eq. (7) now emerge [6]: one can either study boundedness of the

solutions of the initial value problem or undertake a variational analysis. In ideal MHD the first approach has usually been restricted to a study of exponential stability, where a normal-mode time-dependence $e^{i\omega t}$ is assumed. Due to the self-adjointness property the eigenvalues ω^2 will then be real, so that the transition from stability to instability under normal circumstances should occur at $\omega^2 = 0$ (marginal stability). The second approach [5], [7], [8] is to study the sign of the second variation of the potential energy, which in terms of the force-operator reads:

$$W = -\frac{1}{2}\langle \underset{\sim}{\xi}, \underset{\sim}{F}\{\underset{\sim}{\xi}\}\rangle , \qquad (11)$$

where we have dropped variational signs observing that W is just the potential energy of the perturbation $\underset{\sim}{\xi}$.

If the spectrum is discrete and the eigenfunctions of $\underset{\sim}{F}$ form a complete set both approaches are easily shown to lead to the same results. However, in ideal MHD these assumptions are not justified (the common situation being that there is a continuum) and both necessity and sufficiency of the energy principle become much less obvious. The necessity of the energy principle for exponential stability has been re-established [1] by a proof not depending on the completeness property, but the difficulties concerning the sufficiency remain. These difficulties are twofold. Firstly, because the continuum usually extends to the origin the possibility of linear growth arises [6]. Secondly, initial perturbations of p and $\underset{\sim}{B}$ which are not expressible in $\underset{\sim}{\xi}$ (inaccessible initial states, e.g. obtained by breaking of closed field lines) may grow linearly in time even if $W \geq 0$ [9]. The

latter possibility has been excluded in the above-given description, where all perturbed quantities are expressible in ξ (this was meant when we wrote that Eq. (7) provides a "nearly" complete basis for the study of linear stability). However, nothing is known about the time-scale of these linearly growing instabilities. Because we want to improve the usual exponential stability concept rather than the general one, in the following σ-stability analysis the discussion will, therefore, essentially be restricted to exponentially growing instabilities.

We now wish to call an equilibrium σ-stable if there are no perturbations growing faster than $e^{\sigma t}$ and σ-unstable if there are such perturbations. In this way we get around the difficulties connected with non-exponential growth due to the continuous spectrum assuming that these instabilities are not important. Another assumption to be made here is that the unstable side of the spectrum consists of point eigenvalues. This general property of ideal MHD has been conjectured by Grad [2] in analogy with the spectrum of the diffuse linear pinch (see Sec. 3), but a proof is missing. Granted these two assumptions, the extension of the usual exponential stability concept to σ-stability is straightforward.

Definition. An equilibrium is called σ-stable if there exist no point eigenvalues $\omega^2 < -\sigma^2$, and σ-unstable if such eigenvalues exist.

The usual definition of exponential stability is obtained by taking $\sigma = 0$.

Next, a modified energy functional W^σ is introduced, where we absorb in W^σ that part of the kinetic energy which we allow to be present in perturbations:

$$W^\sigma = W + \sigma^2 I \ , \tag{12}$$

where I is the virial

$$I = \frac{1}{2}\langle \underset{\sim}{\xi}, \rho\underset{\sim}{\xi} \rangle \ . \tag{13}$$

This modified energy functional can be written analogously to Eq. (11) in terms of the scalar product with a reduced force-operator $\underset{\sim}{F}^\sigma$:

$$W^\sigma = -\frac{1}{2}\langle \underset{\sim}{\xi}, \underset{\sim}{F}^\sigma\{\underset{\sim}{\xi}\} \rangle \ , \tag{14}$$

where

$$\underset{\sim}{F}^\sigma\{\underset{\sim}{\xi}\} = \underset{\sim}{F}\{\underset{\sim}{\xi}\} - \rho\sigma^2\underset{\sim}{\xi} \ . \tag{15}$$

One could say that the force available for driving a σ-instability is reduced by the amount $\rho\sigma^2\underset{\sim}{\xi}$ with respect to the force available for driving a 0-instability (the usual concept).

For a plasma surrounded by an ideally conducting wall the explicit form of W^σ is

$$W^\sigma = \frac{1}{2}\int [\underset{\sim}{Q}^2 + (\nabla\times\underset{\sim}{B})\cdot\underset{\sim}{\xi}\times\underset{\sim}{Q} + \gamma p(\nabla\cdot\underset{\sim}{\xi})^2 + (\underset{\sim}{\xi}\cdot\nabla p)\nabla\cdot\underset{\sim}{\xi} + \rho\sigma^2\underset{\sim}{\xi}^2]\,d\tau \ . \tag{16}$$

One can also consider a vacuum between the plasma and the wall. In that case, the contribution $\sigma^2 I$ is absorbed in the plasma integral:

$$W^\sigma = W_F^\sigma + W_S + W_V \ , \qquad W_F^\sigma = W_F + \sigma^2 I \ , \tag{17}$$

where W_F, W_S, and W_V are the usual integrals over plasma, interface and vacuum [5].

From the theory of Laval et al. [1] we now obtain the modified energy principle:

An equilibrium is σ-stable if and only if $W^{\sigma}\{\underset{\sim}{\xi}\} > 0$ for all possible $\underset{\sim}{\xi}$.

Introducing the kinetic energy

$$K = \frac{1}{2}\langle \dot{\underset{\sim}{\xi}}, \rho\dot{\underset{\sim}{\xi}} \rangle \quad , \tag{18}$$

and the total energy

$$E = W + K \ , \tag{19}$$

sufficiency and necessity of the modified energy principle are proved from the energy conservation law

$$\dot{E} = 0 \ , \tag{20}$$

and the virial equation

$$\ddot{I} = 2K - 2W \ . \tag{21}$$

The proof of sufficiency is contained in Ref. [1] (one shows that the quantity $K - \sigma^2 I$ cannot grow if $W^{\sigma} > 0$). Necessity is an immediate consequence of the theorem that if $W\{\underset{\sim}{\eta}\} < 0$ for some $\underset{\sim}{\eta}$, then there exists a $\underset{\sim}{\xi}(t)$ growing at least as fast as $e^{\gamma t}$, where $\gamma^2 = -W\{\underset{\sim}{\eta}\}/I\{\underset{\sim}{\eta}\}$. This theorem is proved in Ref. [1] by taking initial values $\underset{\sim}{\xi}(0) = \underset{\sim}{\eta}$, $\dot{\underset{\sim}{\xi}} = \gamma\underset{\sim}{\eta}$, making use of the virial equation (21) and Schwarz's inequality $\dot{I}^2 \le 4IK$. One then demonstrates that $I\{\underset{\sim}{\xi}(t)\} \ge I\{\underset{\sim}{\xi}(0)\}e^{2\gamma t}$, so that $\underset{\sim}{\xi}$ grows at least like $e^{\gamma t}$ and the equilibrium is exponentially unstable. Then, we have $W^{\sigma}\{\underset{\sim}{\eta}\} < 0$ and one can introduce

$$- \frac{W^{\sigma}\{\underset{\sim}{\eta}\}}{I\{\underset{\sim}{\eta}\}} = \lambda^2 > 0 \ , \tag{22}$$

so that

$$\gamma^2 = - \frac{W\{\underset{\sim}{\eta}\}}{I\{\underset{\sim}{\eta}\}} = - \frac{W^{\sigma} - \sigma^2 I}{I} = \lambda^2 + \sigma^2 > \sigma^2 \ . \tag{23}$$

Hence, the equilibrium is not only σ-unstable, but even $\sqrt{\lambda^2 + \sigma^2}$-unstable.

The theory of σ-stability thus turns out to contain the usual theory of exponential stability as a special case. The same limitations with respect to sufficiency of the energy principle apply and it appears that this concept is a useful generalization for practical purposes.

3. The Diffuse Linear Pinch

a) Variational analysis

Minimization of the modified energy functional W^{σ} as expressed in Eq. (14) yields the Euler equation

$$\underset{\sim}{F}^{\sigma}\{\underset{\sim}{\xi}\} = 0 \, . \tag{24}$$

This is actually a system of three coupled partial differential equations. For special equilibria exhibiting a high degree of symmetry one can reduce this system to a system of lower order by taking harmonic solutions corresponding to the ignorable coordinates. We shall here treat the most widely investigated case of the diffuse linear pinch, where the presence of two-fold symmetry allows a reduction to a one-dimensional problem.

Consider an infinitely long straight plasma cylinder surrounded by an ideally conducting wall at $r = a$. This equilibrium is characterized by the profiles $p(r)$, $B_{\theta}(r)$, and $B_z(r)$, which are connected with each other through the equilibrium equation

$$(p + \tfrac{1}{2} B^2)' + B_{\theta}^2/r = 0 \, . \tag{25}$$

We introduce Fourier components of $\underset{\sim}{\xi}$ varying like $e^{i(m\theta + kz)}$ and study each of these components separately, which will not be indicated by further indices. For each Fourier component (m,k) the system of equations (24) then separates out into a second order differential equation for ξ_r and two relations expressing ξ_{θ} and ξ_z in terms of ξ_r and $(r\xi_r)'$. The latter relations are conveniently written in terms of the compressibility and radial component of the flow perpendicular to the lines of force:

$$\nabla\cdot\xi = \frac{\rho\sigma^2}{D}\left[(\rho\sigma^2 + F^2)\,\frac{1}{r}\,(r\xi_r)' - 2\,\frac{kB_\theta}{r}\,\left(\frac{mB_z}{r} - kB_\theta\right)\xi_r\right] , \tag{26}$$

$$\begin{aligned} i(B_\theta\xi_z - B_z\xi_\theta) = \frac{1}{D}\Bigg[&\left(\frac{mB_z}{r} - kB_\theta\right)\{\rho\sigma^2(\gamma p + B^2) + \gamma pF^2\}\,\frac{1}{r}\,(r\xi_r)' \\ &+ 2\,\frac{kB_\theta}{r}\,(\rho\sigma^2B^2 + \gamma pF^2)\xi_r\Bigg] , \end{aligned} \tag{27}$$

where

$$F = mB_\theta/r + kB_z \ , \tag{28}$$

$$D = \rho^2\sigma^4 + \rho\sigma^2(m^2/r^2 + k^2)(\gamma p + B^2) + (m^2/r^2 + k^2)\gamma pF^2 \ . \tag{29}$$

The whole problem thus reduces to solving the second order differential equation for $r\xi_r$:

$$\begin{aligned} &\frac{d}{dr}\left[\frac{(\rho\sigma^2 + F^2)\{\rho\sigma^2(\gamma p + B^2) + \gamma pF^2\}}{D}\,\frac{1}{r}\,\frac{d}{dr}\,(r\xi_r)\right] \\ &\quad - \Bigg[\rho\sigma^2 + F^2 + 2B_\theta\left(\frac{B_\theta}{r}\right)' - 4\,\frac{k^2B_\theta^2}{r^2}\,\frac{\rho\sigma^2B^2 + \gamma pF^2}{D} \\ &\quad + r\left\{2\,\frac{kB_\theta}{r^2}\left(\frac{mB_z}{r} - kB_\theta\right)\frac{\rho\sigma^2(\gamma p + B^2) + \gamma pF^2}{D}\right\}'\Bigg]\xi_r = 0 , \end{aligned} \tag{30}$$

with the boundary conditions

$$r\xi_r = 0 \quad \text{at} \quad r = 0,\ a \ . \tag{31}$$

The obtained reduction is clearly the result of the symmetry of the configuration and the following discussion will, therefore, depend more on the fact that the problem is one-dimensional than on the explicit form of the expressions above. Hence, we introduce the variable $\chi = r\xi_r$ and write the equations (30) and (31) as

$$[f(\sigma)\chi']' - g(\sigma)\chi = 0 \; , \tag{32}$$

$$\chi(r=0) = \chi(r=a) = 0 \; , \tag{33}$$

where the differential equation (32) is just the Euler equation corresponding to the functional

$$J\{\chi\} = \int_0^a [f(\sigma)\chi'^2 + g(\sigma)\chi^2]dr \; . \tag{34}$$

This functional could have been obtained directly from W^σ by writing W^σ in terms of ξ_r, $\nabla \cdot \underset{\sim}{\xi}$, and $i(B_\theta \xi_z - B_z \xi_\theta)$ and rearranging terms such that the latter two components would have appeared as quadratic forms only, leading then to the minimizing expressions (26) and (27). We thus assert that the minimization of the functional $W^\sigma\{\underset{\sim}{\xi}\}$ has been reduced to that of $J\{\chi\}$. This minimization problem is quite similar to the one Newcomb [3] considered in connection with the ordinary energy principle. In fact, we recover his equations if we substitute $\sigma = 0$ in our equations. The crucial difference, however, is the fact that the σ-Euler equation (32) is non-singular for $\sigma^2 > 0$, as is easily seen by inspecting Eq. (30), whereas the marginal Euler equation ($\sigma = 0$) has singularities at the points where $F = 0$. Hence, the variational problem considered here is basically more simple than that of Newcomb and, moreover, concentrates the study on a more significant part of the spectrum. This advantage is of course obtained at the cost of additional terms in the Euler equation, but for numerical computations this represents hardly a complication.

Because the σ-Euler equation is non-singular ($f > 0$) the variational problem is completely standard. Notice that σ is not an

eigenvalue but just a parameter which is fixed beforehand. Therefore, the boundary value problems (32), (33) will have no non-vanishing solutions in general. The calculus of variations (see, e.g., Smirnov [10]) now asserts that the functional $J\{\chi\}$ is positive definite if the solution χ_0 of the Euler equation (32) that vanishes at $r = 0$ does not have a zero in the interval $0 < r \leq a$ (Jacobi's condition). More precisely, Jacobi's condition guarantees that the solution of the Euler equation satisfying the boundary conditions yields a minimum of the integral $J\{\chi\}$. Since the only solution satisfying the boundary conditions is the trivial solution $\chi \equiv 0$, the minimum of $J\{\chi\}$ is zero and $J\{\chi\} > 0$ for all non-vanishing χ satisfying the boundary conditions. The minimum can also be shown to be a strong minimum, i.e., with respect to all trial functions that are close to the minimizing Euler solution as regards the amplitude but not necessarily as regards the derivative (broken curves are allowed). We conclude that Jacobi's condition is both a necessary and sufficient condition for $J\{\chi\} > 0$, and, consequently, for σ-stability. We then have the following theorem for σ-stability of the diffuse linear pinch.

<u>Theorem</u>. <u>For specified values of</u> m <u>and</u> k <u>the diffuse linear pinch is</u> σ-<u>stable if and only if the non-trivial solution of the</u> σ-<u>Euler equation</u> (32) <u>that vanishes at</u> $r = 0$ <u>does not have a zero in the open interval</u> $(0, a)$.

In order to obtain Newcomb's results from this theorem one only has to show that in the limit $\sigma \to 0$ the solutions of the Euler equation to the left and to the right of singular points ($F = 0$)

decouple, so that the interval is split into independent subintervals. This has been demonstrated elsewhere [11].

We have to report one complication of the use of the σ-stability concept for the case of the diffuse linear pinch. In the usual marginal stability analysis one has the useful property that a pinch is stable for all m and k if and only if it is stable for $m = 0$, $k \to 0$ and $m = 1$, all k (Theorem 1 of Ref. [3]). One can, therefore, restrict a marginal stability analysis to the cases $m = 0$ and $m = 1$. For the σ-stability analysis one can derive a similar theorem employing the functional dependence on σ, m, and k of the functions f and g appearing in the Eqs. (30), (32), and (34). Denoting this dependence as $f(\sigma,m,k)$, $g(\sigma,m,k)$, and $W^\sigma(m,k)$ one easily derives the following properties. For $m = 0$:

$$f(\sigma, 0, |k_o|) = f(|k/k_o|\sigma, 0, |k|) \ ,$$

$$g(\sigma, 0, |k_o|) \leq g(|k/k_o|\sigma, 0, |k|) \ , \quad \text{for} \quad |k| \geq |k_o| \ ,$$

so that

$$W^\sigma(0, |k_o|) \leq W^{|k/k_o|\sigma}(0, |k|) \ , \quad \text{for} \quad |k| \geq |k_o| \ .$$

For $m \neq 0$:

$$f(\sigma, 1, k) = f(|m|\sigma, |m|, |m|k) \ ,$$

$$g(\sigma, 1, k) \leq g(|m|\sigma, |m|, |m|k) \ , \quad \text{for} \quad |m| \geq 1 \ ,$$

so that

$$W^\sigma(1, k) \leq W^{|m|\sigma}(|m|, |m|k) \ , \quad \text{for} \quad |m| \geq 1 \ .$$

Hence, if a linear pinch is σ-stable for $m = 0$, $|k| = |k_o|$, it is

$(|k/k_0|\sigma)$-stable for $m = 0$, $|k| \geq |k_0|$; if the pinch is σ-stable for $m = 1$, all k, it is $(|m|\sigma)$-stable for $|m| \geq 1$. This theorem reduces to the old result in the limit $\sigma \to 0$, but it is of little practical use because one would have to allow very large values of σ in order to assure σ-stability for large values of m and k. Moreover, the theorem cannot be reversed. It does not imply that the linear pinch is always σ-unstable for sufficiently large values of m and k. This would conflict with the proof of Hain et al. [7] that growth rates of MHD instabilities are bounded from above if the plasma density is bounded from below. The bounds on the growth rates given in Ref. [7] are also of little practical use because they are large and independent of any details of the equilibrium. We conclude that one has to examine all values of m and k in a σ-stability analysis.

b) Connection with the spectral analysis (Sturmian property for $\omega^2 < 0$)

The σ-stability analysis is directly related to the spectral analysis of the unstable side of the spectrum. This is evident from the fact that the Euler equation (24) is identical with the equation of motion for normal modes with a time-dependence $e^{i\omega t}$ if one replaces σ^2 by $-\omega^2$. Accordingly, the same substitution transforms the σ-Euler equation (30) into the equation of motion for the diffuse linear pinch as derived by Hain and Lüst [12]. In order to distinguish between the two equations and their solutions we shall label them with σ and $i\omega$, respectively. The eigenvalue problem to be solved in order to determine eigenvalues ω for the diffuse linear pinch is thus written as

$$[f(i\omega)\chi']' - g(i\omega)\chi = 0 \ , \tag{35}$$

$$\chi(r=0) = \chi(r=a) = 0 \ , \tag{36}$$

where $f(i\omega)$ and $g(i\omega)$ are given by the expressions of Eq. (30) replacing σ^2 by $-\omega^2$. It is clear from the way in which ω^2 appears in f and g that the eigenvalue equation (35) is non-singular for $\omega^2 < 0$, so that the unstable side of the spectrum is a point spectrum. For $\omega^2 = 0$ the equation transforms into the singular marginal stability equation. For $\omega^2 > 0$ the equation develops a number of singularities, which give rise to singular eigenfunctions and continuous spectra. It belongs to the deep mysteries of the plasma physics literature that well over a hundred papers have been written on the stability of the diffuse linear pinch, but that up till recently [13]-[16], [2], [17], none of these studies employed the basic equation of motion derived by Hain and Lüst in 1958 [12], whereas the presence and consequences of a continuous spectrum hardly received attention let alone a proper treatment. Notice that the σ-stability analysis avoids a study of the continuous spectrum by definition.

Although the equations (35), (36) are identical with the equations (32), (33), the problem to be solved is quite different. In the σ-stability analysis σ is fixed and the boundary value problem (32), (33) has no solution in general. In order to obtain the σ-stability condition the boundary value problem is then replaced by the initial value problem $\chi(r=0) = 0$, $\chi'(r=0) = 1$ and one has to integrate the equation (32) once over the interval $(0,a)$ in order to

find out whether there is a zero (σ-unstable) or not (σ-stable). In the spectral analysis ω has to be determined from the eigenvalue problem (35), (36) which is mathematically more complicated to the extent that a boundary value problem is more complicated than an initial value problem. This is especially of importance for the numerical work which is reported in the following paper. For the numerical determination of σ-stability one integration of Eq. (32) from $r = 0$ to $r = a$ suffices, whereas the calculation of eigenvalues requires a number of integrations. On the other hand, a numerical program for the investigation of σ-stability is easily converted into one which actually calculates eigenvalues. It thus appears that the σ-stability analysis is a rather optimum combination of the standard stability analysis based on the energy principle and a normal-mode analysis, avoiding the mathematical difficulties of the first and reducing the amount of work needed in the second approach.

The connection between σ-stability and the unstable side of the spectrum of the diffuse linear pinch is based on the Sturmian property of the equation of motion (35) for $\omega^2 < 0$. By this is meant the property that the zeros of oscillatory solutions monotonically move away from each other upon increase of $-\omega^2$. Therefore, if we have a solution $\chi(\sigma)$ of Eq. (32) satisfying the boundary condition $\chi(r = 0) = 0$ and vanishing at least once in the interval $(0, a)$ we conclude that for $\omega^2 < -\sigma^2$ a solution $\chi(i\omega)$ of Eq. (35) exists satisfying both boundary conditions (36), so that the system is σ-unstable (Fig. 1a). If the solution $\chi(\sigma)$ has no zero (Fig. 1b) no such eigenfunction exists and the system is σ-stable. It is

clear that the Sturmian property is the basic property of the equation of motion of the diffuse linear pinch for $\omega^2 < 0$ from which the σ-stability theorem of Sec. 3a follows immediately. We devote the remainder of this section to a proof of this property.

For $\omega^2 < 0$ the equation (35) is non-singular, but contains the eigenvalue ω^2 in a complicated manner. If one could demonstrate monotonicity of f and g as a function of ω^2 Sturmian behavior of the discrete spectrum would be guaranteed by Sturm's fundamental theorem with the modification of Picone (see Ince [18]). However, the problem is not of this standard type because the function g is monotonic in ω^2 only for small and large values of $-\omega^2$, but for intermediate values of $-\omega^2$ the derivative in curly brackets in Eq. (30) spoils the monotonicity completely. Nevertheless, Sturmian behavior can be proved quite generally for one-dimensional systems because the operator $\tilde{F}$ of the equation of motion is self-adjoint, so that ω^2 is real. We first present a proof depending solely on this fact and not on the specific form of the coefficients f and g. Having obtained this proof the question then naturally arises how the Sturmian behavior derives from the properties of the functions f and g. Therefore, we also present a second proof where the specific form of the differential equation enters.

Since the proofs require only a minor modification to include the stable side of the spectrum also, we present a slightly more general discussion than is needed here, including stable point spectra. This extension was motivated by the conjecture of Grad [2] that the discrete spectrum as a whole, including the stable side, is non-Sturmian and by the discovery

of points of accumulation on the stable side by Tataronis [19]. We shall show here that the discrete spectrum consists of a number of subspectra separated by continua where each discrete subspectrum is either Sturmian or anti-Sturmian and has at most one point of accumulation at the edge of the subspectrum.

c) Sturmian and anti-Sturmian discrete spectra.

The eigenvalue equation $(f\chi')' - g\chi = 0$ develops a number of singularities for $\omega^2 \geq 0$ which is most clearly seen by writing

$$f = \frac{\gamma p + B^2}{r} \; \frac{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}{(\omega^2 - \omega_3^2)(\omega^2 - \omega_4^2)} \;,$$

where

$$\omega_1^2\,(r) = F^2/\rho \;,$$

$$\omega_1^2\,(r) = \frac{\gamma p}{\gamma p + B^2}\, F^2/\rho \;, \tag{37}$$

$$\omega_{3,4}^2(r) = [\tfrac{1}{2}(m^2/r^2+k^2)(\gamma p+B^2) \pm \tfrac{1}{2}\sqrt{(m^2/r^2+k^2)^2(\gamma p+B^2)^2-4(m^2/r^2+k^2)\gamma p F^2}\,]/\rho \;. \tag{38}$$

For fixed values of $r \neq 0$ these four local frequencies are well-ordered according to

if $0 < F^2 < (m^2/r^2+k^2)B^2$: $\quad \omega_2^2 < \omega_3^2 < \omega_1^2 < \omega_4^2$,

if $F^2 = 0$: $\quad 0 = \omega_2^2 = \omega_3^2 = \omega_1^2 < \omega_4^2 \neq 0$,

if $F^2 = (m^2/r^2+k^2)B^2$: $\quad \begin{cases} \omega_2^2 < \omega_3^2 = \omega_1^2 < \omega_4^2 \; (\text{for } \gamma p > B^2) \\ \omega_2^2 < \omega_3^2 < \omega_1^2 = \omega_4^2 \; (\text{for } \gamma p < B^2) \;. \end{cases}$

On the interval $0 \leq r \leq a$ the functions $\omega_2^2(r)$, $\omega_3^2(r)$, $\omega_1^2(r)$, and $\omega_4^2(r)$ take continuous values in ranges which we will designate by $\{\omega_2^2\}$, $\{\omega_3^2\}$, $\{\omega_1^2\}$, $\{\omega_4^2\}$, where, e.g., $\{\omega_2^2\}$ indicates $\omega_{2\ \min}^2 \leq \omega^2 \leq \omega_{2\ \max}^2$. These ranges of ω^2 usually correspond to continuous spectra [2], [19] which may or may not overlap depending on the specific equilibrium profiles and the values of m and k. E.g., if the interval $0 \leq r \leq a$ contains a point where $F = 0$ the first three continua overlap and extend to the origin $\omega^2 = 0$. In that case only the fourth continuum may occur separated from the rest. If there is no point $F = 0$ the four continua may all occur separated if the axis $r = 0$ is excluded (e.g. by considering a hard-core pinch). If the axis is included the continua $\{\omega_3^2\}$ and $\{\omega_2^2\}$ always overlap because $\omega_3^2(0) = \omega_2^2(0)$, whereas for $m \neq 0$ the continuum $\{\omega_4^2\}$ extends up to infinity because of the singularity m^2/r^2.

We now consider the real ω^2-axis omitting the singular regions $\{\omega_2^2\}$, $\{\omega_3^2\}$, $\{\omega_1^2\}$, $\{\omega_4^2\}$ so that we obtain subintervals of the real ω^2-axis where the eigenvalue equation (35) is non-singular (except for the singularity at $r = 0$) and f has a definite sign. Consider such a subinterval and suppose that there is a point spectrum for values of ω^2 lying in that subinterval. This implies that the solutions χ of Eq. (35) are oscillatory. We then prove that in that subinterval the number of zeros of eigenfunctions χ is a monotonic function of ω^2, i.e. either monotonically increasing (classical Sturmian behavior) or decreasing (anti-Sturmian behavior).

First proof

Consider the interval $0 \leq r \leq a$ and solve Eq. (35) for fixed m and k and some value ω^2 in the chosen ω^2-subinterval, subject to the left boundary condition $\chi(r = 0) = 0$. The latter condition, incidentally, takes care of the only remaining singularity of Eq.(35), viz. the one occurring at $r = 0$. Because we assumed the presence of a point-spectrum, we thus obtain a well-behaved solution χ having a finite number of zeros on $0 < r \leq a$. Consequently, we find a discrete number of possible wall positions $0 < R \leq a$ for which the boundery value problem would be solved. For fixed ω^2 this function R is by definition an increasing function of n, the number of zeros of the solution χ in $0 < r < a$.

Now consider the functional dependence of R on ω^2 in the horizontal strip $0 < R \leq a$ in the R-ω^2 plane, from which the vertical strips $\omega^2 = \{\omega_2^2\}$, $\{\omega_3^2\}$, $\{\omega_1^2\}$, $\{\omega_4^2\}$ are omitted (Fig. 2, which shows for purposes of illustration a special case where neither of the continua overlap or extend up to infinity, which can only be realized in a hard-core pinch). If ω^2 is in the range of possible pointeigenvalues, to each ω^2 there corresponds at least one value of $R \leq a$ and different values of R corresponding to the same value of ω^2 are labelled by n. We prove that the function $R = R(n,\omega^2)$ is a monotonic function of ω^2 on each of the branches n within the same non-singular ω^2-subinterval. Notice that these branches cannot cross because this would lead to a contradiction with the fact that R is an increasing function of n for fixed ω^2.

Suppose that an extremum occurred for some value of R and ω^2, say $R = R_o$ and $\omega^2 = \omega_o^2$, then one could write

$$R - R_o \approx \frac{1}{2} \left[\frac{\partial^2 R}{\partial(\omega^2)^2}\right]_o [\delta(\omega^2)]^2 . \tag{39}$$

This relation gives the real function R of the real argument ω^2 in the neighborhood of $\omega^2 = \omega_o^2$, where R is the position of a real zero of the solution χ of Eq. (35). However, since f and g are non-singular functions of r and ω^2 for $0 < r \leq a$ and values of ω^2 in one non-singular subinterval, we may as well restrict the equilibrium profiles to those which make f and g analytic functions of both r and ω^2. The function $R = R(n,\omega^2)$ will then be analytic in ω^2 and the expansion (39) can be analytically continued in the complex ω^2-plane. Then, one can choose $[\delta(\omega^2)]^2 < 0$ so that $\delta(\omega^2)$ is imaginary and, consequently, $\omega^2 \approx \omega_o^2 + \delta(\omega^2)$ is complex. Equation (39) then gives for this complex ω^2 a possible real wall-position for which the boundary value problem $\chi = 0$ at $r = 0$ and $r = R$ would be solved. This contradicts the fact that ω^2 is real for self-adjcint systems and we conclude that an extremum for R as a function of ω^2 cannot occur. Hence, in each non-singular ω^2-subinterval R is a monotonic function of ω^2 on each of the branches labelled by n.

Having proved that for fixed ω^2 R is an increasing function of n, and for fixed n R is a monotonic function of ω^2, it follows that for fixed R ω^2 is a monotonic function of n. Choosing $R = a$, we see that the solution ω^2 of the boundary value problem (35) is a monotonic function of n. This proves that the discrete spectra away from the continua are either Sturmian ($\partial R/\partial\omega^2 < 0$) or anti-Sturmian ($\partial R/\partial\omega^2 > 0$). For the unstable side of the spectrum we have seen already that Eq. (35) is of a standard type for small and large values of $-\omega^2$. This determines the sign of the

monotonicity and we find that the discrete spectrum for $\omega^2 < 0$ (or, more general, $\omega^2 < \omega^2_{2\ min}$) is Sturmian.

For the present purpose the Sturmian property of the unstable side of the spectrum is all that is needed because it shows that the maximum growth rate occurs for n = 0, the zero-node solution (see Fig. 2). However, we now also wish to determine the sign of the monotonicity for the discrete spectra occurring on the stable side. This might be established by developing Eq. (35) in the neighborhood of the edges of the discrete spectra. Instead, it is more instructive to turn to the second proof which gives the sense of the monotonicity automatically and, moreover, offers some insight into the physical origin of the non-standard anti-Sturmian behavior which is encountered on the stable side of the spectrum.

Second proof.

We have seen that the usual proof of Sturmian behavior for Eq. (35) breaks down because the function g contains a derivative with respect to r of a function of ω^2. This term arises in the course of the reduction of the original three-dimensional eigenvalue problem to the equivalent one-dimensional one given by the Eqs. (35) and (36). It is precisely this reduction which obsures the original self-adjointness property of the system and causes the breakdown of the classical correspondence between orthogonality of the eigenfunctions and number of nodes. We, therefore, return to the original three-dimensional system and employ the self-adjointness explicitly.

The self-adjointness of the ideal MHD force-operator $\underset{\sim}{F}(\underset{\sim}{\xi})$ stems

from the fact that the expression

$$\underset{\sim}{\eta}\cdot\underset{\sim}{F}\{\underset{\sim}{\xi}\} - \underset{\sim}{\xi}\cdot\underset{\sim}{F}\{\underset{\sim}{\eta}\}$$

can be written as a divergence, viz.:

$$\begin{aligned}\underset{\sim}{\eta}\cdot\underset{\sim}{F}\{\underset{\sim}{\xi}\} - \underset{\sim}{\xi}\cdot\underset{\sim}{F}\{\underset{\sim}{\eta}\} = \nabla\cdot[\underset{\sim}{\eta}\{&\underset{\sim}{\xi}\cdot\nabla p + \gamma p\nabla\cdot\underset{\sim}{\xi} - \underset{\sim}{B}\cdot\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B})\} \\ &- \underset{\sim}{\xi}\{\underset{\sim}{\eta}\cdot\nabla p + \gamma p\nabla\cdot\underset{\sim}{\eta} - \underset{\sim}{B}\cdot\nabla\times(\underset{\sim}{\eta}\times\underset{\sim}{B})\} \\ &+ \underset{\sim}{B}\{\underset{\sim}{\eta}\cdot\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B}) - \underset{\sim}{\xi}\cdot\nabla\times(\underset{\sim}{\eta}\times\underset{\sim}{B}) - (\nabla\times\underset{\sim}{B})\cdot(\underset{\sim}{\eta}\times\underset{\sim}{\xi})\}].\end{aligned} \tag{40}$$

For equilibria with B.n = 0 self-adjointness follows immediately from the application of Gauss' theorem to this equation for regular vector fields $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ satisfying the boundary conditions $\underset{\sim}{\xi}\cdot\underset{\sim}{n} = \underset{\sim}{\eta}\cdot\underset{\sim}{n} = 0$. Here, we drop the latter condition and obtain the following expression:

$$\begin{aligned}&\int[\underset{\sim}{\eta}\cdot\underset{\sim}{F}\{\underset{\sim}{\xi}\} - \underset{\sim}{\xi}\cdot\underset{\sim}{F}\{\underset{\sim}{\eta}\}]d\tau \\ &\quad = \int[\underset{\sim}{\eta}\cdot\underset{\sim}{n}\{\gamma p\nabla\cdot\underset{\sim}{\xi} - \underset{\sim}{B}\cdot\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B})\} - \underset{\sim}{\xi}\cdot\underset{\sim}{n}\{\gamma p\nabla\cdot\underset{\sim}{\eta} - \underset{\sim}{B}\cdot\nabla\times(\underset{\sim}{\eta}\times\underset{\sim}{B})\}]d\sigma,\end{aligned} \tag{41}$$

where the terms with ∇p have been cancelled out because we consider only equilibria with $\nabla p \parallel \underset{\sim}{n}$.

Consider two solutions $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ of the equation of motion corresponding to certain values of ω^2, say ω_I^2 and ω_{II}^2:

$$\begin{aligned}\underset{\sim}{F}\{\underset{\sim}{\xi}\} &= -\rho\omega_I^2\,\underset{\sim}{\xi}\ , \\ \underset{\sim}{F}\{\underset{\sim}{\eta}\} &= -\rho\omega_{II}^2\,\underset{\sim}{\eta}\ .\end{aligned} \tag{42}$$

Let $\underset{\sim}{\xi}$ be an eigenfunction, but $\underset{\sim}{\eta}$ not. Eq. (41) then transforms to:

$$\int \underset{\sim}{\eta}\cdot\underset{\sim}{n}\ [\gamma p\nabla\cdot\underset{\sim}{\xi} - \underset{\sim}{B}\cdot\nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B})]d\sigma = -\ (\omega_I^2 - \omega_{II}^2)\int\rho\underset{\sim}{\xi}\cdot\underset{\sim}{\eta}\ d\tau. \qquad (43)$$

This expression is still general. We now turn to one-dimensional problems, particularly the diffuse linear pinch.

Introducing Fourier components.

$$\underset{\sim}{\xi} = (\ \hat{\xi}_{r,mk}(r),\ \hat{\xi}_{\theta,mk}(r),\ \hat{\xi}_{z,mk}(r))\ e^{i(m\theta+kz)}, \qquad (44)$$

or rather real parts of such expressions since we use a real-type scalar product, all three-dimensional expressions can be reduced to terms involving the radial components of $\underset{\sim}{\eta}$ and $\underset{\sim}{\xi}$ only. We drop the circumflex and indices m and k again and understand by $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ expressions of the above form. The essential point in the derivation is now to reduce the left-hand side of Eq.(43) to a one-dimensional form, but to retain the three-dimensional form for the right-hand side. To that end we employ the expressions (26) and (27), replacing σ^2 by $-\omega^2$, and obtain by straightforward algebra:

$$2\pi L\ [r\eta_r\ f_I(r\xi_r)'\ ]_{r=a} \ - 2\pi L\ [r\eta_r\ f_I(r\xi_r)'\ ]_{r=0}$$

$$= -\ (\omega_I^2 - \omega_{II}^2)\int\rho\underset{\sim}{\xi}\cdot\underset{\sim}{\eta}d\tau, \qquad (45)$$

where $f_I = f(\omega_I^2)$ as given by Eq. (37) and $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ refer to the same mode numbers m and k. The second term on the left-hand side is due to the singularity of the equation of motion on the axis, which has to be excluded in the integration of Eq. (43). We drop this contribution again by imposing the left-hand boundary

condition on η, $r\,\eta_r(r = 0) = 0$, so that we obtain:

$$[r\eta_r\, f_I(r\xi_r)']_{r=a} = -(\omega_I^2 - \omega_{II}^2) \int \rho \underset{\sim}{\xi} \cdot \underset{\sim}{\eta}\; r\, dr. \qquad (46)$$

This expression holds for all values ω^2 away from the singular regions $\{\omega_2^2\}$, $\{\omega_3^2\}$, $\{\omega_1^2\}$, $\{\omega_4^2\}$.

Let $r = 0$ and $r = a$ be two subsequent zeros of $r\xi_r$, i.e. consider the zero-node solution, and choose $r\xi_r > 0$ on $0 < r < a$ so that $(r\xi_r)'_{r=a} < 0$. Choose ω_I^2 and ω_{II}^2 in the same non-singular subinterval and close enough that $\underset{\sim}{\eta}$ is close to $\underset{\sim}{\xi}$, so that $\int \rho \underset{\sim}{\xi} \cdot \underset{\sim}{\eta}\, d\tau > 0$. Suppose that $f_I > 0$, that $\omega_{II}^2 > \omega_I^2$, and that $r\eta_r$ does not vanish on $0 < r \leq a$. Because $\underset{\sim}{\eta}$ is close to $\underset{\sim}{\xi}$ the latter condition implies that $r\eta_r > 0$ on $0 < r \leq a$. Under these conditions the left-hand side of Eq. (46) is negative, whereas the right-hand side is positive This contradiction proves that $r\eta_r$ has to vanish at a point in between the two zeros of $r\xi_r$. This is just the Sturmian property for the zero-node solution. The higher-node solutions can be treated in the same way by considering the subintervals of $0 < r \leq a$ between zeros of $r\xi_r$. We conclude that a non-singular discrete subspectrum where $f > 0$ is Sturmian. On the other hand, the subspectra where $f < 0$ are anti-Sturmian, i.e. the number of nodes of the eigenfunctions diminishes upon increase of ω^2. This property is proved along the same lines reversing the roles of $\underset{\sim}{\eta}$ and $\underset{\sim}{\xi}$.

Therefore, the complete spectrum consists of Sturmian and anti-Sturmian non-singular point-spectra separated by continuous spectra. For the case of maximum separation of the subspectra, depicted in

Fig. 2, there are at most four continua and five point spectra, three of which are Sturmian and two anti-Sturmian. A further analysis of the equation of motion shows that the points $\omega^2_{2\ min}$, $\omega^2_{2\ max}$, $\omega^2_{1\ min}$, $\omega^2_{1\ max}$, and ∞ may be points of accumulation, so that each non-singular point spectrum is bounded by at most one point of accumulation, in agreement with the Sturmian and anti-Sturmian properties of the point spectra. In the respresentation of Fig. 2 a point of accumulation would show up as an accumulation point of the intersections of infinitely many branches $R(n,\omega^2)$ with the vertical edge bounding a continuum. An example of such behavior is shown in Fig. 3 where R^2 is plotted as a function of $-\omega^2$ for the different $m = 1$ pure interchange instabilities in a constant-pitch magnetic field [16]. This case is even more peculiar than the general case just described because the accumulating intersections collapse into the point $R = 0$, whereas simultaneously three of the four continua contract into the origin $\omega^2 = 0$.

4. Suydam's criterion

The marginal stability analysis of the diffuse linear pinch in an equilibrium with shear leads to an Euler equation which is singular at the points $r = r_s$ where the function F of Eq. (28) vanishes. An expansion of the Euler equation around such points yields Suydam's [20] criterion

$$p' + \frac{1}{8} r B_z^2 \left(\frac{\mu'}{\mu}\right)^2 > 0 , \tag{47}$$

which is a necessary condition for 0-stability of the diffuse linear pinch with finite shear. This expansion breaks down upon the introduction of terms with σ in the Euler equation so that Suydam's criterion loses its sense in a σ-stability analysis. One can violate it and still find σ-stable equilibria. This is particularly evident in low-shear systems which approximate closed-line systems. For these systems Suydam's criterion has been shown to be irrelevant because its violation only implies negligible growth whereas different semi-local stability criteria acquire more importance [2]. From these observations one could conclude that Suydam's criterion can be safely ignored in a σ-stability analysis. This section will be largely devoted to showing the contrary for intermediate and high-shear systems. From now on, we, therefore, restrict our discussion to these systems.

Suydam's criterion is both a necessary and sufficient condition for stability of modes which are sufficiently localized in directions perpendicular to the magnetic field. Such a localization can be obtained by means of a superposition of modes with a large number of nodes in radial, azimuthal, and longitudinal directions, i.e.,

for large values of n, m, and k. Since m and k are coupled through the condition $k_{||} = k + \mu m = 0$ (there is no localization along the field lines), there are actually only two numbers characterizing a Suydam mode, e.g., n and m. In a strict sense, Suydam's criterion refers to the transition from stability to instability of modes for which both n and m tend to infinity. It would appear that these modes are not particularly significant in ideal MHD since they are certainly stabilized by finite Larmor radius effects and, therefore, they are not very well described by ideal MHD. However, the significance of Suydam's criterion does not reside in the stability of these modes, but in derived properties for the less localized lower-node solutions.

The least localized modes are, in general, the $m = 1$, $n = 0$ modes, i.e., modes with a minimum number of nodes in radial (n) and azimuthal (m) direction (excluding the less important $m = 0$ modes for the present). The remarkable feature of Suydam's criterion is now that, while it is actually the stability condition for the $m \to \infty$, $n \to \infty$ modes, its violation implies that the entire infinity of all lower-node solutions, including $m = 1$, $n = 0$, is unstable. [Of course, we are now considering only modes for which $k = -\mu(r_s)m$.] That $m = 1$ is unstable if $m \to \infty$ is unstable is an immediate consequence of the monotonicity of W as a function of m, as shown by Newcomb [3]. Also, the usual derivation of Suydam's criterion shows that it is a necessary criterion for the stability of all modes for which $k + \mu(r_s)m = 0$, no matter what the value of m is. On the other hand, that $n = 0$ is unstable if $n \to \infty$ is unstable is an immediate consequence of the Sturmian property proved in the preceding section. As we have seen, this property provides the connection between the marginal stability analysis and the spectral analysis.

For the case of Suydam modes the marginal equation of motion reduces to the confluent hypergeometric equation [21] having a spiral point at $x = r - r_s = 0$. Hence, the solution of the marginal equation of motion vanishes at a curve in the complex r-plane spiralling around the point $r = r_s$. The Sturmian property of the equation of motion for $\omega^2 < 0$ now implies that the spiral unwinds if $-\omega^2$ is increased, starting from $\omega^2 = 0$. In this way the number of zeros of $r\xi_r$ on the real axis decreases and eigenvalues are obtained each time that the spiral crosses both points $r = 0$ and $r = a$. Consequently, for fixed m the highest growth rate is obtained for the mode with the least localization, i.e., $n = 0$. This conclusion only holds with respect to localization in radial direction, the dependence of the growth rate on the localization in azimuthal direction (m) being quite different.

In Figs. 4-8 the numerical results are shown of a computation of eigenfunctions and growth rates of an equilibrium that is Suydam unstable. We have chosen a particular equilibrium that is Suydam stable on axis but which has a large region where Suydam's criterion is violated (Fig. 4). In Fig. 5 the different eigenfunctions are shown for $m = 1$ and an increasing number of nodes in radial direction. The sequence starts with the zero-node solution which is hardly localized, apart from the fact that it has a maximum in the region where Suydam's criterion is violated. Increasing the number n of nodes, the modes become more and more localized around the point r_s where $k + \mu(r_s)m = 0$. This localization manifests itself in two ways. Firstly, because of the increasing number of zeros close to $r = r_s$, one can build up an extremely localized perturbation in radial direction

by a suitable superposition of these modes. Secondly, because the energy of a perturbation involves both ξ_r and ξ_r', for each of the modes separately the bulk of the energy is localized close to $r = r_s$ where ξ_r has large derivatives (i.e., ξ_θ and ξ_z are large). A similar sequence of eigenfunctions for fixed n (n = 0) and increasing m is shown in Fig. 6. Because of the azimuthal dependence $e^{im\theta}$ it is also possible to construct a highly localized wave packet in azimuthal direction by a superposition of the higher m modes. Such a wave packet would be localized in radial direction as well as is evident from the increasing radial localization with increasing m shown in Fig. 6.

The dependence of the growth rate on n and m is shown in Figs. 7 and 8. The growth rate is a decreasing function of n, as it should be according to the Sturmian property. Moreover, this dependence turns out to be exponential, in excellent agreement with analytical results obtained by Grad [2] and Pao [22]. As a result, the first mentioned construction in the preceding paragraph of a radially localized m = 1 perturbation by the superposition of a large number of modes with different n, does not make too much sense. Because the constituent modes have completely different growth rates, the initially localized form of the perturbation will be immediately destroyed and after some time the non-localized m = 1, n = 0 mode will completely dominate. In Fig. 8 the growth rate is plotted of the zero-node solutions as a function of m for fixed k/m (the dotted lines) and for the value of k where $-\omega^2$ is maximum (the solid line). Comparing Figs. 7 and 8 one sees that localization in azimuthal direction has a quite different effect on the growth rate than localization in radial direction. Because the

curve becomes flat for higher values of m, it is clear that the last-mentioned construction in the preceding paragraph of a localized wave packet consisting of a large number of $n = 0$ modes with different values of m has a definite physical meaning. Such a wave packet conserves its localized shape and grows exponentially just as fast as every single constituent mode does. For the construction of it one should of course take modes with fixed values of $k/m = -\mu(r_s)$, i.e., centered at $r = r_s$. For single modes, on the other hand, the maximum growth rate as indicated by the solid line in Fig. 8 is more significant.

We conclude that violation of Suydam's criterion implies that not only highly localized perturbations around a single line of force are unstable, but also gross $m = 1$, $n = 0$ modes. The first type of perturbations leads to local mixing of the equilibrium profiles, possibly resulting in enhanced diffusion, whereas the second type of perturbations leads to the usual kink-like loss of plasma. Although the localized perturbations may grow faster, their effect is less immediate and they are also much more subject to stabilizing finite Larmor-radius effects. For these reasons, the real importance of Suydam's criterion in ideal MHD is given by the fact that it provides a necessary stability condition for the gross $m = 1$, $n = 0$ modes.

Returning now to the σ-stability analysis, we finally have to consider the numerical values of the growth rates in order to decide from a practical point of view whether Suydam's criterion has to be taken seriously or not in a σ-stability analysis. In the given example the $m = 1$, $n = 0$ mode has a growth rate given by

$\Omega^2 = \rho\omega^2 a^2/B_o^2 = -5.31 \times 10^{-3}$ which corresponds to an e-folding time $\tau = 2.4\ \mu$ sec for the case of a hydrogen plasma with 10^{15} particles per cm^3, a tube radius of 50 cm and magnetic field of 100 kG. This is a huge growth rate which cannot be tolerated in any thermonuclear confinement experiment. It is true that this is just one result from one particular equilibrium that is Suydam unstable. We will report extensively about a large number of different equilibria in the accompanying paper. The evidence abstracted from these numerical computations confirms that the here reported feature is fairly general, i.e., that violation of Suydam's criterion for intermediate and high-shear systems leads to gross $m = 1$, $n = 0$ modes with appreciable growth rates. Therefore, in a σ-stability analysis of these systems it is wise, though not always necessary, to observe Suydam's criterion.

Another piece of evidence from numerical work concerns the dependence of the growth rate on m. In Sec. 3a we had to conclude that the convenient theorem of Newcomb, stating that it is sufficient to consider $m = 0$ and $m = 1$ only in order to be able to conclude about stability for all modes, reduces to a far less useful result in the σ-stability analysis. Here, from σ-stability of the $m = 1$ modes one can only conclude that the pinch is at least $|m|\sigma$-stable for arbitrary m. Figure 8 is a particular example showing why this result is obtained. Initially the growth rate can increase with increasing m and the theorem gives an upper bound of the growth rate just extrapolating this initial increase. This extrapolation is obviously far too pessimistic and misses completely the point that the growth rate has to be bounded. If Suydam's criterion is satisfied unstable modes occur only below a certain value of m,

because for $m \rightarrow \infty$ the pinch is then stable (Suydam's criterion is a sufficient stability criterion for $m \rightarrow \infty$). Usually, the growth rate is then a decreasing function of m, so that σ-stability only has to be tested for the lower-m modes, such as $m = 0,1,2,3,\ldots$. For the systematic search of σ-stable high-shear equilibria undertaken in the accompanying paper, Suydam's criterion thus serves the very useful dual purpose of reducing both the number of possible equilibria and the number of modes to be tested for σ-stability.

Acknowledgements

The authors are indebted to Professors Harold Grad, Harold Weitzner, and Drs. Günther Spies and John Tataronis for useful discussions and suggestions.

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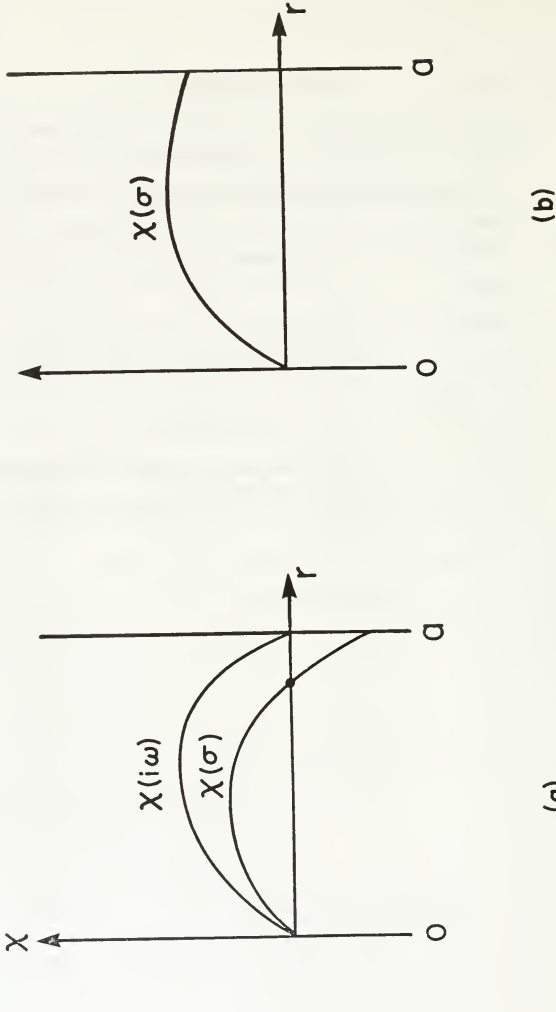

Fig. 1. Solutions of the σ-Euler equation.
a) σ-unstable ($\omega^2 < -\sigma^2$); b) σ-stable.

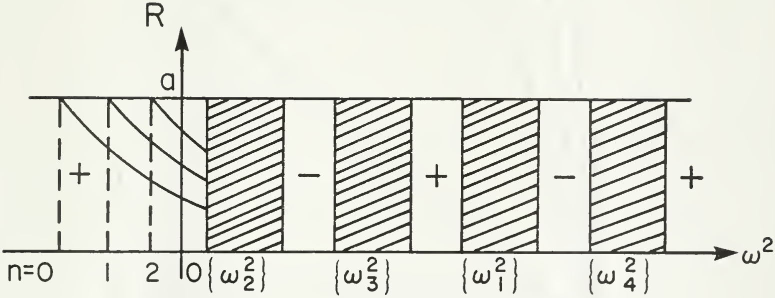

Fig. 2. Sturmian and anti-Sturmian point-spectra

+ : Sturmian, - : anti-Sturmian.

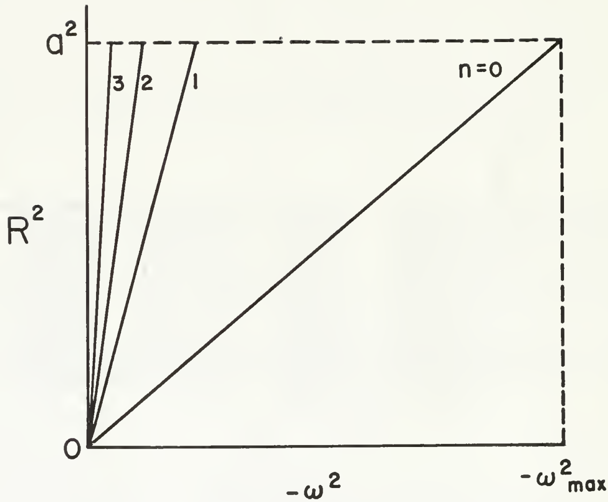

Fig. 3. Sturmian property for pure interchanges in a constant-pitch magnetic field.

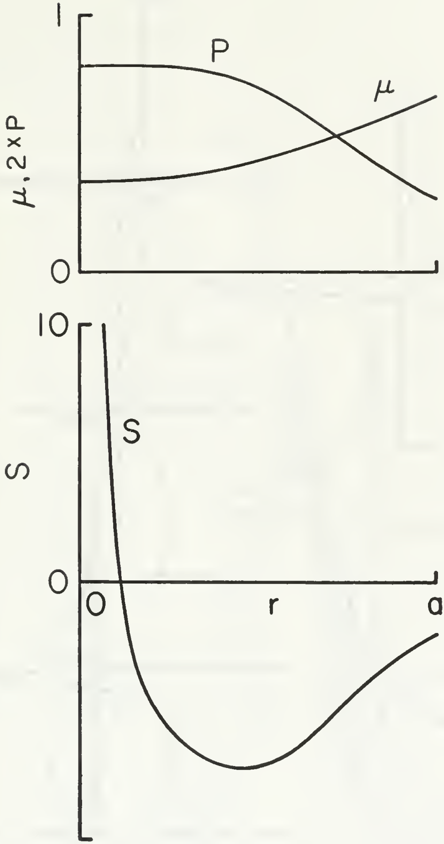

Fig. 4. Equilibrium profiles and Suydam function
$S(r) = 1 + \frac{8p'}{rB_2^2}(\frac{\mu}{\mu'})^2.$

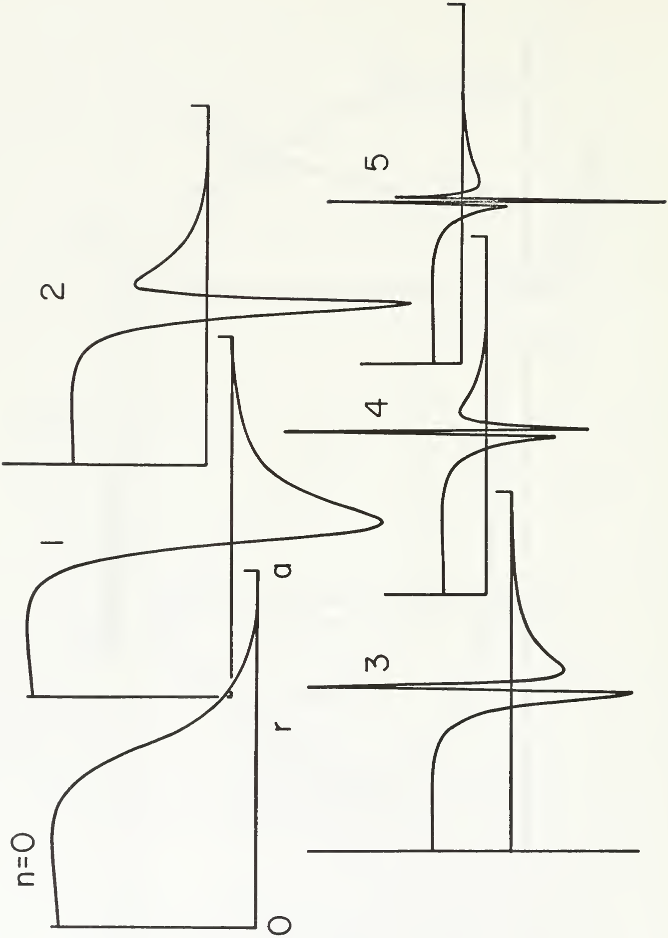

Fig. 5. m = 1 Suydam modes for increasing number of nodes.

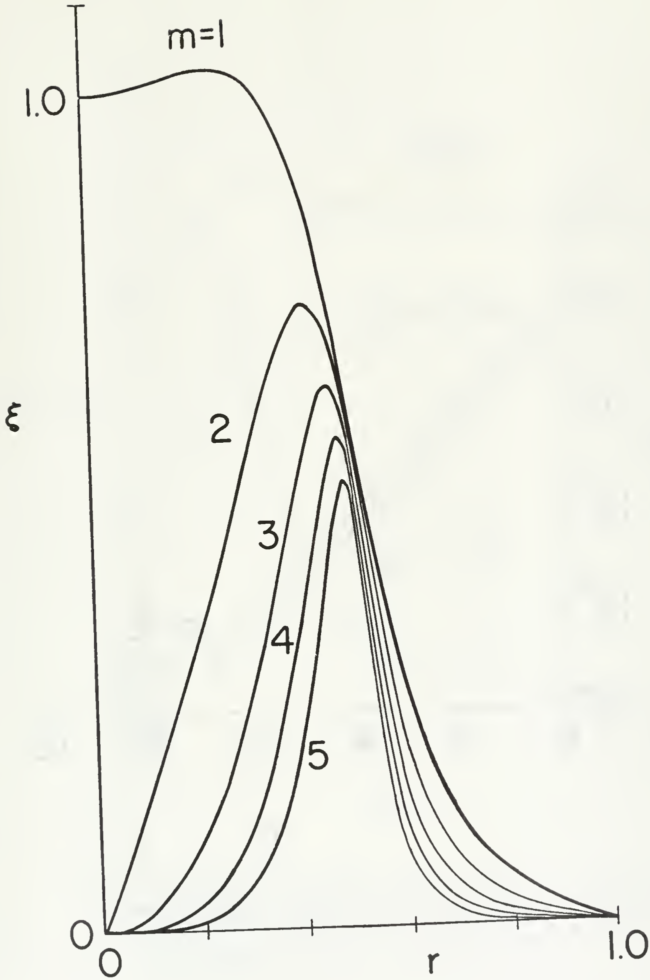

Fig. 6. n = 0 Suydam modes for increasing m .

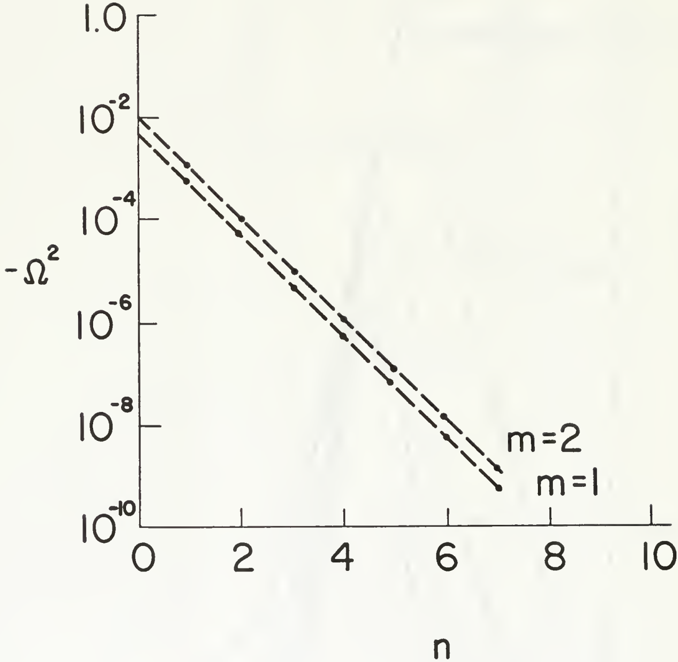

Fig. 7. Growth rate of Suydam modes versus n, for m = 1 and 2 modes.

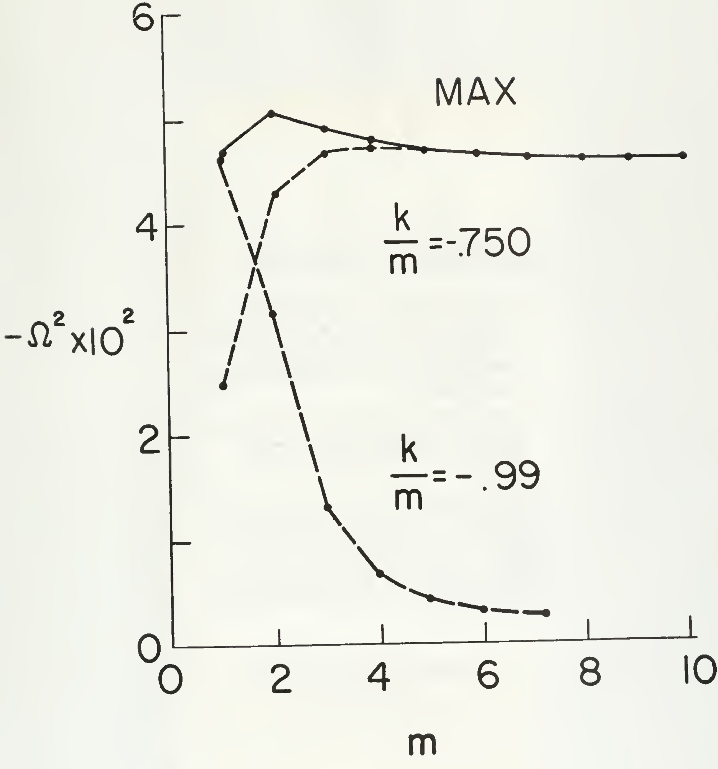

Fig. 8. Growth rate of Suydam modes versus m, for fixed k/m (the dotted lines) and for the values of k where $-\omega^2$ is maximum (the solid line).

COO-3077-29 c.1

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